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A CENTRAL LIMIT THEOREM FOR NUMBERS SATISFYING A CLASS OF TRIANGULAR ARRAYS

A. KYRIAKOUSSIS

Statistical Unit, Department of Mathematics, University of Athens, Panepistemiopolis, Athens 621, Greece

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A central limit theorem for the numbers $A(m, n) \geq 0$, satisfying a class of triangular arrays, is established. Several combinatorial applications are discussed.

1. Introduction and notation

We suppose that the numbers $A(m, n)$ satisfy the following class of triangular arrays.

$$A(m+1, n) = [a_1(m)n + a_0(m)]A(m, n) + [b_1(m)n + b_0(m)]A(m, n-1), \tag{1.1}$$

$$n = 0, 1, \ldots, m+1, \qquad m = 0, 1, \ldots, \quad \text{if } a_0(0) \neq 0,$$
$$n = 1, 2, \ldots, m+1, \qquad m = 0, 1, \ldots, \quad \text{if } a_0(0) = 0,$$

with boundary conditions

$$A(0, 0) = 1, \qquad A(0, n) = 0 \quad \text{if } n \neq 0, \tag{1.2}$$

and with $\{a_1(m)n + a_0(m)\}$ and $\{b_1(m)n + b_0(m)\}$ positive for the corresponding values of n and m.

With the numbers $A(m, n)$ we associate the combinatorial distribution

$$P_m(n) = A(m, n)\lambda^n / A_m(\lambda), \quad n = 0, 1, \ldots, m, \qquad \lambda > 0, \tag{1.3}$$

where

$$A_m(\lambda) = \sum_{j=0}^{m} A(m, j)\lambda^j. \tag{1.4}$$

We say that $A(m, n)$ satisfy a central limit theorem or they are asymptotically normal with mean μ_m and variance σ_m^2 if

$$\lim_{m\to\infty} \sup_x \left| \sum_{n \leq \sigma_m x + \mu_m} p_m(n) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp[-\tfrac{1}{2}t^2]\,dt \right| = 0. \tag{1.5}$$

Several authors studied the asymptotic normality for certain numbers $A(m, n)$, which are special cases of (1.1). Among them, Harper [4] for $A(m, n)$ the Stirling numbers of the second kind, Charalambides [2], for $A(m, n)$ the C-numbers, Tanny [8] for $A(m, n)$ the Eulerian numbers, etc.

In the present paper we prove that (1.5) holds when $a_1(m)$, $b_1(m)$ are constant functions and $a_0(m)$, $b_0(m)$ are linear functions in m (see Corollary 2.1) and when $a_1(m)$, $b_1(m)$, $a_0(m)$, $b_0(m)$ are 'multiples' of a common function of m (see Corollary 2.2). Moreover, this paper covers in particular the above known results and most standard cases of triangular arrays.

2. A central limit theorem

Let

$$\varphi_m(s)=\sum_{n=0}^{m} A(m, n)\lambda^n e^{sn}. \tag{2.1}$$

Theorem 2.1. *If*

$$\varphi_m(s)=a(m)[r(s)]^{m-1}A(s), \tag{2.2}$$

where $r(s)$ and $A(s)$ have bounded third derivative near zero,

$$[r'(0)/r(0)]^2-r''(0)/r(0)\neq 0, \qquad A(0)\neq 0,$$

and

$$a(m)\neq 0, \quad m=0,1,2,\ldots, \qquad a(0)=1,$$

then (1.5) *holds with*

$$\mu_m=-(m+1)[r'(0)/r(0)]+A'(0)/A(0)$$

and

$$\sigma_m^2=(m+1)\{[r'(0)/r(0)]^2-r''(0)/r(0)\} + A''(0)/A(0)-[A'(0)/A(0)]^2.$$

Proof. We consider the characteristic function of $p_m(n)$. For convenience shift the mean by μ_m and the variance by σ_m^2 and call this function $f_m(t)$. Since

$$\sum_{n=0}^{m} A(m, n)\lambda^n e^{inx}=\varphi_m(ix),$$

it follows that

$$f_m(t)=\exp[-i\mu_m t/\sigma_m]\varphi_m(it/\sigma_m)/\varphi_m(0) = \exp\{-i\mu_m t/\sigma_m+\ln[\varphi_m(it/\sigma_m)/\varphi_m(0)]\}.$$

Since
$$\mu_m=\varphi_m'(0)/\varphi_m(0)=-(m+1)[r'(0)/r(0)]+A'(0)/A(0),$$
$$\begin{aligned}\sigma_m^2&=\varphi_m''(0)/\varphi_m(0)-[\varphi_m'(0)/\varphi_m(0)]^2\\&=(m+1)\{[r'(0)/r(0)]^2-r''(0)/r(0)\}\\&\quad+A''(0)/A(0)-[A'(0)/A(0)]^2\to\infty\quad\text{as } m\to\infty,\end{aligned}$$
and $r(s)$ and $A(s)$ have bounded third derivative near 0 we have, by expanding $\ln[\varphi_m(\mathrm{i}t/\sigma_m)/\varphi_m(0)]$ in a Taylor series,
$$f_m(t)=\exp[-\tfrac{1}{2}t^2+O(t^3/m^{\frac{1}{2}})],$$
uniformly for sufficiently small t/σ_m.

So
$$f_m(t)\sim\exp[-\tfrac{1}{2}t^2]$$
for all t. By the continuity theorem [5, Section 13, Theorem 2] and [5, Section 9] we have the validity of (1.5). □

Corollary 2.1. *If $a_1(m)=c_2$, $a_0(m)=c_3m+c_4$, $b_1(m)=c_5$, $b_0(m)=c_6m+c_7$, then $A(m,n)$ are asymptotically normal with μ_m and σ_m^2 as in Theorem 2.1, where $r(s)$, $A(s)$ are the solutions of the differential equations*
$$\begin{aligned}&-(c_2+\lambda e^sc_5)r'(s)+(c_3+\lambda e^sc_6)r(s)=c_1,\quad c_1 \text{ constant},\\&(c_2+\lambda e^sc_5)A'(s)+[\lambda e^s(c_7+c_5-c_6)-c_3+c_4]A(s)=0,\\&[r'(0)/r(0)]^2-r''(0)/r(0)\neq0,\ A(0)\neq0.\end{aligned}\tag{2.3}$$

Proof. By (1.1), (2.1) and our assumptions we obtain
$$\varphi_{m+1}(s)=(c_2+e^sc_5)\varphi_m'(s)+[c_3m+c_4+\lambda e^s(c_6m+c_7+c_5)]\varphi_m(s),\tag{2.4}$$
where $\varphi_m'(s)=\mathrm{d}\varphi_m(s)/\mathrm{d}s$ and $\varphi_0(s)=1$.

Choosing $a(m)=c_1^m m!$, $m=0,1,2,\ldots$, and $r(s)$, $A(s)$ the solutions of (2.3), we can easily verify that $\varphi_m(s)=a(m)[r(s)]^{m-1}A(s)$ is the solution of the difference-differential equation (2.4). Theorem 2.1 is applied. □

Corollary 2.2. *If there is a function $\beta(m)$ such that $\beta(m)\neq0$ $m=0,1,2,\ldots$, and $a_1(m)/\beta(m)=c_9$, $b_1(m)/\beta(m)=c_{10}$, $a_0(m)/\beta(m)=c_{11}$, $[b_0(m)+b_1(m)]/\beta(m)=c_{12}$, then $A(m,n)$ are asymptotically normal with μ_m and σ_m^2 as in Theorem 2.1, where $r(s)$, $A(s)$ are the solutions of the differential equations*
$$\begin{aligned}&(c_9+\lambda e^sc_{10})r'(s)+c_8=0,\quad c_8 \text{ constant},\\&(c_9+\lambda e^sc_{10})A'(s)+(c_{11}+\lambda e^sc_{12})A(s)=0,\\&[r'(0)/r(0)]^2-r''(0)/r(0)\neq0,\ A(0)\neq0.\end{aligned}\tag{2.5}$$

Proof. By (1.1), (2.1) and our assumptions we obtain
$$\varphi_{m+1}(s)=(c_9+\lambda e^sc_{10})\beta(m)\varphi_m'(s)+(c_{11}+\lambda e^sc_{12})\beta(m)\varphi_m(s).\tag{2.6}$$

where
$$\varphi'_m(s)=\mathrm{d}\varphi_m(s)/\mathrm{d}s, \qquad \varphi_0(s)=1.$$

Choosing $a(m)=(c_8)^m\beta(m-1)\cdots\beta(0)$, $m=1,2,\ldots$, $a(0)=1$ and $r(s)$, $A(s)$ the solutions of (2.5), we can easily verify that $\varphi_m(s)=a(m)[r(s)]^{-m-1}A(s)$ is the solution of (2.6). Theorem 2.1 is applied. □

Remark. By (1.1) and (2.1) we obtain

$$\varphi_{m+1}(s)=[a_1(m)+\lambda e^s b_1(m)]\varphi'_m(s)+[a_0(m)+\lambda e^s\{b_0(m)+b_1(m)\}]\varphi_m(s), \tag{2.7}$$

where

$$\varphi'_m(s)=\mathrm{d}\varphi_m(s)/\mathrm{d}s, \qquad \varphi_0(s)=1.$$

In Corollaries 2.1 and 2.2 our goal was to find conditions on $a_0(m)$, $b_0(m)$, $a_1(m)$, $b_1(m)$ under which there are functions $a(m)$, $r(s)$, $A(s)$ such that $\varphi_m(s)$, given by (2.2), is the solution of (2.7).

The equation (2.7) on using (2.2) is equivalent with

$$\begin{aligned}\frac{a(m+1)}{a(m)}A(s)=&-(m+1)[a_1(m)+\lambda e^s b_1(m)]r'(s)A(s)\\&+[a_1(m)+\lambda e^s b_1(m)]r(s)A'(s)\\&+[a_0(m)+\lambda e^s\{b_0(m)+b_1(m)\}]r(s)A(s).\end{aligned} \tag{2.8}$$

From (2.8) it follows that there are many ways to find conditions on $a_0(m)$, $b_0(m)$, $a_1(m)$, $b_1(m)$ with the same as above result. For example, if $a_1(m)/\gamma(m)=l_2$, $b_1(m)/\gamma(m)=l_3$, $a_0(m)/(m+1)\gamma(m)=l_4$ and $[b_0(m)+b_1(m)]/(m+1)\gamma(m)=l_5$, for some $\gamma(m)\neq 0$ $m=0,1,2,\ldots$, l_i, $i=2,3,4,5$, constants, then (2.8) is verified for $a(m)=l_1^m\gamma(m-1)\cdots\gamma(0)$, l_1 constant and $r(s)$, $A(s)$ the solution of the system

$$(l_2+\lambda e^s l_3)r'(s)-(l_4+\lambda e^s l_5)r(s)=-l_1, \qquad (l_2+\lambda e^s l_3)r(s)A'(s)=0.$$

But we don't know any special case of (1.1) with $a_0(m)\neq 0$ satisfying such as the above conditions.

3. Applications

(1) When $A(m,n)$ are numbers associated with Stirling numbers of the first kind, (see [1]), we have $A(m+1,n)=(2m+1-n)(A(m,n-1)+A(m,n))$. Consequently, $a_1(m)=b_1(m)=-1$, $a_0(m)=b_0(m)=2m+1$. In addition $c_2=c_5=-1$, $c_3=c_6=2$, $c_4=c_7=1$ and the solution of (2.3) is $r(s)=(c_1/\lambda^2)[\lambda e^{-s}-e^{-2s}\ln(1+\lambda e^s)-ce^{-2s}]$, c constant, $A(s)=e^{-s}(1+\lambda e^s)^{-1}$.

So, by Corollary 2.1 we have that (1.5) holds with

$$\begin{aligned}\mu_m=(m+1)[2\lambda-2\ln(1+\lambda)-\lambda^2/(1+\lambda)-2c]/[\lambda-\ln(1+\lambda)-c]\\-1/(1+\lambda)-2\lambda/(1+\lambda)\end{aligned}$$

and

$$\sigma_m^2=(m+1)\{([2\lambda-2\ln(1+\lambda)-\lambda^2/(1+\lambda)-2c]/[\lambda-\ln(1+\lambda)-c])^2$$
$$-[4\lambda-4\ln(1+\lambda)-4\lambda^2/(1+\lambda)+(\lambda+2)\lambda^2/(\lambda+1)-4c]/[\lambda-\ln(1+\lambda)-c]\}$$
$$+1/(1+\lambda)-2\lambda(1+2\lambda)/(1+\lambda)^2-[(1+2\lambda)/(1+\lambda)]^2.$$

(2) When $A(m,n)$ are numbers associated with Stirling numbers of the second kind (see [1]), we have $A(m+1,n)=(2m+1-n)A(m,n)+(m+1-n)A(m,n-1)$. Consequently, $a_1(m)=b_1(m)=-1$, $a_0(m)=2m+1$, $b_0(m)=m+1$. In addition $-c_2=c_4=-c_5=c_6=c_7=1$, $c_3=2$ and the solution of (2.3) is

$$r(s)=e^{-2s}(c_1/\lambda^2)[(1+\lambda e^s)\ln(1+\lambda e^s)-\lambda ce^s+1-c] \quad\text{and}\quad A(s)=e^{-s}$$

So, by Corollary 2.1 we have that (1.5) holds with

$$\mu_m=(m+1)\{(2+\lambda)[\ln(1+\lambda)+(1+\lambda)^{-1}]-\lambda^2(1+\lambda)^{-1}$$
$$+c(1+\lambda)\}/\{(1+\lambda)\ln(1+\lambda)+1-c-c\lambda\}-1$$

and

$$\sigma_m^2=(m+1)\left\{\left(\frac{(2+\lambda)[\ln(1+\lambda)+(1+\lambda)^{-1}]-\lambda^2(1+\lambda)^{-1}+c+c\lambda}{(1+\lambda)\ln(1+\lambda)+1-c-c\lambda}\right)^2\right.$$
$$-\frac{(4+\lambda)[\ln(1+\lambda)+(1+\lambda)^{-1}]-(2+\lambda)[\lambda/(1+\lambda)-\lambda(1+\lambda)^{-2}]-\lambda^3(1+\lambda)^{-2}-c(2+\lambda)}{(1+\lambda)\ln(1+\lambda)+1-c-c\lambda}$$

(3) When $A(m,n)$ are the cumulative numbers (see [3]), we have $A(m+1,n)=(n+a)A(m,n)+(m-n+2-a)A(m,n-1)$. Consequently, $a_1(m)=-b_1(m)=1$, $a_0(m)=a$, $b_0(m)=m+2-a$. So we have $c_2=-c_5=c_6=1$, $c_3=0$, $c_4=a$, $c_7=2-a$ and the solution of (2.3) is

$$r(s)=\begin{cases}(1-\lambda e^s)^{-1}(-c_1s+c), & c\neq 0 \quad \text{if } s\neq\ln(1/\lambda)\\ c_1 & \text{if } s=\ln(1/\lambda)\end{cases} \quad\text{and}\quad A(s)=e^{-as}.$$

So, by Corollary 2.1 we have that (1.5) holds with

$$\mu_m=(m+1)[c_1/c-\lambda/(\lambda-1)]-a \quad \text{if } \lambda\neq 1$$

and

$$\sigma_m^2=(m+1)[(c_1/c)^2-\lambda/(\lambda-1)^2] \quad \text{if } \lambda\neq 1.$$

In the case that $\lambda=1$ from (2.6) we have $r(0)=c_1$, $r'(0)=-\frac{1}{2}c_1$, $r''(0)=\frac{1}{6}c_1$ and consequently $\mu_m=\frac{1}{2}(m+1)-a$ and $\sigma_m^2=\frac{1}{12}(m+1)$. For $a=0$ we have the well-known Eulerian numbers for which Tanny [8] proved that they are asymptotically normal ($\lambda=1$), by relating them with the distribution of the sum of uniform random variables on $[0,1)$.

(4) When $A(m,n)$ are the non-central Stirling numbers of the second kind (see [6]), we have $A(m+1,n)=(n+a)A(m,n)+A(m,n-1)$, $a\geqslant 0$. Consequently,

$a_1(m)=1$, $b_1(m)=0$, $a_0(m)=a$, $b_0(m)=1$. So, we have $c_2=c_7=1$, $c_3=c_5=c_6=0$, $c_4=a$ and the solution of (2.3) is $r(s)=-c_1s+c$, $c\neq 0$ and $A(s)=\exp[-a\lambda-\lambda e^s]$. So, by Corollary (2.1) we have that (1.5) holds with

$$\mu_m=(m+1)(c_1/c)-a-\lambda \quad \text{and} \quad \sigma_m^2=(m+1)(c_1/c)^2-\lambda.$$

For $a=0$ we have the usual Stirling numbers of the second kind for which Harper [4] derived for $\lambda=1$ similar conclusions using a more complicated method.

(5) When $A(m,n)$ are Whitney's numbers for a class of geometric lattices (see [7]), we have $A(m+1,n)=(ln+1)A(m,n)+A(m,n-1)$, $l>0$. In this case, $a_1(m)=l$, $b_1(m)=0$, $a_0(m)=b_0(m)=1$. So, we have $c_2=l$, $c_3=c_5=c_6=0$, $c_4=c_7=1$ and the solution of (2.3) is

$$r(s)=(-c_1/l)s+c,\ c\neq 0 \quad \text{and} \quad A(s)=\exp[(-1/l)s-(1/l)\lambda e^s].$$

So, by Corollary 2.1 we have that (1.5) holds with

$$\mu_m=(m+1)[c_1/(lc)]-1/l-\lambda \quad \text{and} \quad \sigma_m^2=(m+1)[c_1/(lc)]^2-\lambda.$$

(6) When $A(m,n)=\binom{m}{n}$ we have $\varphi_m(s)=(1+\lambda e^s)^m$. Theorem 2.1 is applied with $r(s)=A(s)=(1+\lambda e^s)^{-1}$. So, (1.5) holds with $\mu_m=m\lambda/(1+\lambda)$ and $\sigma_m^2=m\lambda/(1+\lambda)^2$. We note that (1.3) is the binomial distribution with $p=\lambda/(1+\lambda)$.

(7) When $A(m,n)$ are the C-numbers (see [2]), we have $A(m+1,n)=(ln-m)A(m,n)+lA(m,n-1)$, $l>1$. So, $a_1(m)=b_0(m)=l$, $a_0(m)=-m$, $b_1(m)=0$. Since $ln-m>0$ for $n=1,2,\dots,m+1$, $m=0,1,2,\dots$ we must have $m\leq l$. Consequently, $l\to\infty$ as $m\to\infty$. Setting $m/l=a$, a constant, such that $0<a<1$, the conditions of Corollary 2.2 are satisfied with $\beta(m)=m$. In addition $c_9=c_{12}=1/a$, $c_{10}=0$, $c_{11}=-1$ and the solution of (2.3) is

$$r(s)=-ac_8s+c, \qquad c\neq 0 \quad \text{and} \quad A(s)=\exp[as-\lambda e^s/a].$$

So, (1.5) holds with $\mu_m=(m+1)ac_8/c+a-\lambda/a$ and $\sigma_m^2=(m+1)(ac_8/c)^2-\lambda/a$. Similarly we can prove that (1.5) holds for the $|C(m,n,-l)|$ numbers, $l>1$, (see [2]).

References

[1] L. Carlitz, Note on the numbers of Jordan and Ward, Duke Math. J. 38 (1971) 783–790.

[2] Ch. Charalambides, The asymptotic normality of certain combinatorial distributions, Ann. Inst. Statist. Math. 28 (1976) 499–506.

[3] Dwyer, The cumulative numbers and their polynomials, Ann. Math. Statist. 11 (1940) 66–71.

[4] L.H. Harper, Stirling behavior is asymptotically normal, Ann. Math. Statist. 38 (1967) 410–414.

[5] B.V. Gnedenko and A.N. Kolmogorov, Limit Distributions for Sums of Independent Random Variables (trans. from Russian by K.L. Chung) (Addison-Wesley, Cambridge, MA, 1954).

[6] M. Koutras, Non-central Stirling numbers and some applications, Discrete Math. 42 (1982) 73–89.

[7] J. Stonefisher, Logarithmic concavity for a class of geometric lattices, J. Combin. Theory, Ser. A 18–19 (1975) 216–218.

[8] S. Tanny, A probabilistic interpretation of Eulerian numbers, Duke Math. J. 40 (1973) 717–722.